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LADDER MODELS FOR THE CONSTITUTIVE BEHAVIOR OF HETEROGENEOUS MATERIALS WITH DAMAGE

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Abstract—Many materials which are used in engineering applications, such as fiber reinforced plastic composites, metal matrix composites and also more traditional materials such as concrete, belong to a large class of multiphase, inhomogeneous solids. When these components are used for structural applications, the evaluation of structural safety and durability can only be carried out by performing numerical analyses which use a representative constitutive law of the homogenized medium, partially losing a detailed description of the microstructure. Naturally, the better constitutive models are those retaining a larger number of relevant features present in the behavior at the microstructural level, though this attention for detail cannot be pushed beyond a limit which is assigned by the random variations in the microgeometry. It is the intent of this work to describe a procedure in which simplified constitutive models for random composites can be defined from the mechanical behavior of each component. The derivation of the overall properties does not rely on a linear elastic analysis of the microstructure but, as it is done in a more refined way in finite element studies, the parameters governing the interaction between the different phases are obtained from a purely topological description of the material. In the second part of this work, attention is devoted to the constitutive behavior of concrete, which is a binary composite with random distribution of phases. Some of the features characterizing the softening response of this material are incorporated in a new constitutive model for the matrix phase (mortar). Finally, some results are given for a binary composite incorporating a softening and a linearly elastic phase for two simple loading histories.

1. INTRODUCTION

The search for a representative constitutive law for a composite solid has been characterized by several contributions; for the case where the different phases are randomly distributed, a well known tool of analysis is the theory of mixtures (Green and Naghdi, 1965), which will be addressed in more detail in the next section. Other theories of homogenization or related work go from the early studies on elliptical inclusions (Eshelby, 1957) to Mori and Tanaka (1973) and similar models (e.g. Benveniste, 1987) and are all characterized by an initial assumption; the behavior of the multiphase material is captured by a constitutive model containing parameters which are obtained from a linear analysis of the microstructure. This initial hypothesis somewhat impairs the effectiveness of all constitutive laws which are subsequently derived. At the other end of the scale one finds the finite element studies in which the microstructure can be fully represented with the greatest detail, but from which it is difficult to extract general laws and, furthermore, with which it is difficult to cope with the statistical variations of distribution, size and shape at the microstructural level. In particular, if one takes a rather common material, for example concrete, the range of grain dimensions for the dispersed phase (aggregate) can vary from tenths of a millimeter to a few centimeters; no finite element study can possibly capture the complete interaction between the smaller and the larger material scales given the enormous amount of discretization which would be required. An aspect of the theory presented in this work, as it will be more carefully explained in the next section, is the possibility of taking into account the average interaction between the different material scales. This objective can be accomplished by considering a mechanical model containing building elements, connected in some way, which have very different representative volumes. One can therefore build a

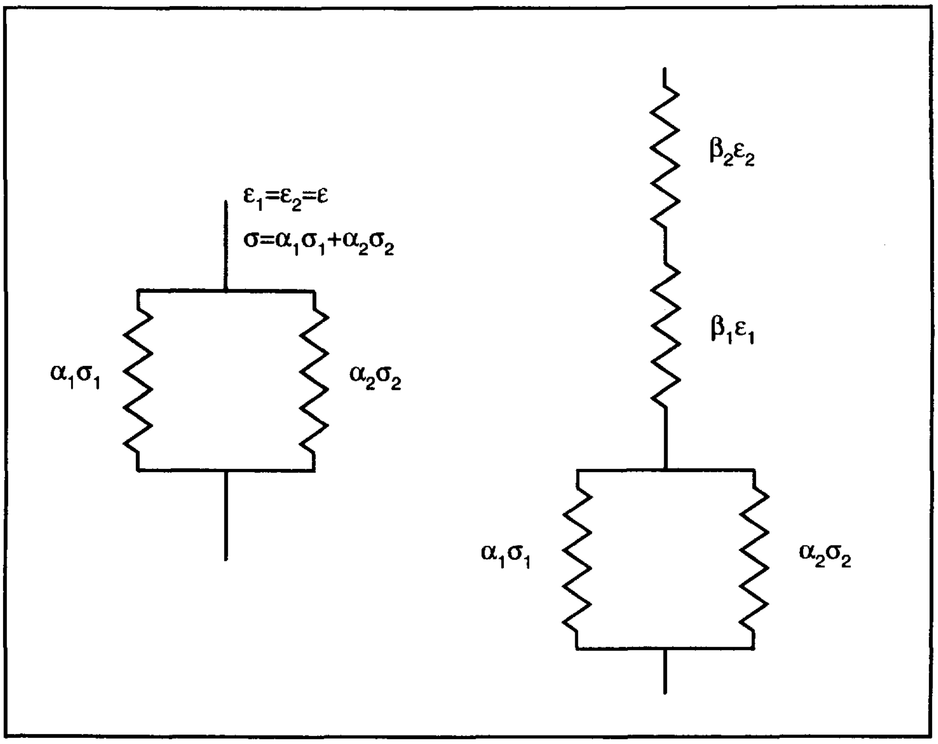

Fig. 1. A mechanical representation of the constitutive behavior predicted by the theory of mixtures for a binary composite (left); a model with more degrees of freedom is shown on the right.

mechanical equivalent of the composite under scrutiny which spans across a few levels of micromechanics.

2. CONSTITUTIVE MODELS FOR HETEROGENEOUS SOLIDS

2.1. *The theory of mixtures*

Among the simplified constitutive models which serve as computational tools for the analysis of complex problems, the first one which comes to mind for simplicity and widespread use is the theory of mixtures. Such a theory considers a representative continuum where the stiffness of each phase contributes to the overall one in a measure directly proportional to its volume fraction. If $\mathbf{K}_i$ is the stiffness† of each phase and y_i the corresponding volume fraction, the stiffness of the composite material is given by

$$\mathbf{K}_{\text{overall}} = \sum \mathbf{K}_i y_i, \tag{1}$$

and each phase is considered to be subjected to the same strain as the overall composite. A pictorial representation of the assumptions of the theory of mixtures is shown in Fig. 1(a) for a two-phase material; the parallel assembly of the two phases reflects the additivity of the material stiffnesses and the equivalence between the phase strains.

Although this is a very simplified way of representing the constitutive behavior of a composite solid, there is one aspect of this theory which needs to be highlighted because it will also appear in more complex material models. If one considers two phases characterized by having different stiffnesses, one can show that a uniaxial state of stress will produce a compressive stress in one phase and a tensile stress in the other along directions which are normal to the axis of loading. For example, if the value of Poisson's ratio is not the same in each phase, the two isolated phases would respond to uniaxial loading with different lateral contractions or elongations. Since every component of the strain tensor has to be

† Throughout this study, boldface variables denote matrices or tensors. In addition, when the equations are written in component form, the indicial notation and conventional sum over repeated indices are used.

the same in each phase of the composite material, some additional internal stresses will arise so that the state of deformation will be equal. From the point of view of the composite, these internal phase stresses will not appear since their net sum, weighted by the volume fractions of each phase, has to vanish. Nonetheless, they will have to be considered whenever the state of stress of each individual component is evaluated, for example, when yield conditions are taken into account.

2.2. *Ladder models for binary composites*

One can consider the theory of mixtures as a starting point for the construction of more refined material representations. In particular, a visual examination of any cross section of a random composite reveals that, for a given axis of loading, some fractions of the different phases are indeed reacting in parallel to the external action while others are assembled in a manner that more closely resembles a series of deformable entities. As a first modification of the first order mixture theory, one could consider a series assembly of two representative elements, as shown in Fig. 1(b); one of them is made up of the two phases working in parallel, the other one of the same phases reacting in series. The state of stress will be different in each single component and it will be derived assuming that each part of material contributes to the overall response in measure directly proportional to its volume fraction. For this purpose it is useful to introduce the concept of "finite" stiffness $\mathbf{D}$ and "finite" compliance $\mathbf{D}^{-1}$, defined by considering the series and parallel assembly of two finite volumes of material. In the parallel assembly of elastic elements, as shown in Fig. 1(a), the contribution of each phase to the overall stiffness is directly proportional to the corresponding volume as in the first order theory of mixtures. The overall finite stiffness is then given by

$$\mathbf{D}_{\text{overall}} = V_1\mathbf{K}_1 + V_2\mathbf{K}_2 \tag{2}$$

where $\mathbf{K}_1, \mathbf{K}_2$ are the constitutive matrices of the phases and V_1 and V_2 are the corresponding volumes. In terms of the volume fractions $\alpha_i = V_i/V_{\text{ref}}$, V_{ref} being the total or some reference volume, one writes the finite stiffness as

$$\mathbf{D}_{\text{overall}}^{\text{parallel}} = V_{\text{ref}}(\alpha_1\mathbf{K}_1 + \alpha_2\mathbf{K}_2). \tag{3}$$

The above law, valid also in the case where the sum of the volume fractions is not unity, can then be used to obtain the constitutive stiffness $\mathbf{K}$ as the volume average

$$\mathbf{K}_{\text{overall}}^{\text{parallel}} = \frac{1}{V_1+V_2}\mathbf{D}_{\text{overall}}^{\text{parallel}} = \frac{1}{\alpha_1+\alpha_2}(\alpha_1\mathbf{K}_1 + \alpha_2\mathbf{K}_2). \tag{4}$$

Accordingly, the finite compliance $\mathbf{D}^{-1}$ of a series assembly of two phases of volume fractions β_1 and β_2 can be derived as

$$\mathbf{D}_{\text{overall}}^{-1} = V_{\text{ref}}(\beta_1\mathbf{K}_1^{-1} + \beta_2\mathbf{K}_2^{-1}) \tag{5}$$

or, after derivation of the constitutive compliance and inversion,

$$\mathbf{K}_{\text{overall}}^{\text{series}} = (\beta_1+\beta_2)(\beta_1\mathbf{K}_1^{-1} + \beta_2\mathbf{K}_2^{-1})^{-1}. \tag{6}$$

It is straightforward to notice that, for any value of the volume fractions, the above formulaes render an unchanged stiffness if both phases are set to correspond to the same material. If one now considers a binary composite represented by an assembly of four elastic elements as shown in Fig. 1(b) (β_1 and β_2 being the volume fractions of the two phases working in series and α_1 and α_2 the corresponding values for the parallel assembly), one can compute the overall constitutive stiffness as

$$\mathbf{K}_{\text{overall}} = (\beta_1+\beta_2+\alpha_1+\alpha_2)\,[\beta_1(\mathbf{K}_1)^{-1}+\beta_2(\mathbf{K}_2)^{-1}+(\alpha_1+\alpha_2)^2(\alpha_1\mathbf{K}_1+\alpha_2\mathbf{K}_2)^{-1}]^{-1}. \quad (7)$$

Again, if the two phases have the same elastic response, the above relation renders the unchanged material stiffness. The overall strain will be given by

$$\boldsymbol{\varepsilon} = \mathbf{K}_{\text{overall}}\boldsymbol{\sigma}. \quad (8)$$

If one distinguishes the entities pertaining to the four different elements with the subscripts β_1, β_2, α_1 and α_2, one can write the overall strain as

$$\boldsymbol{\varepsilon} = \frac{1}{\beta_1+\beta_2+\alpha_1+\alpha_2}[\beta_1\boldsymbol{\varepsilon}_{\beta_1}+\beta_2\boldsymbol{\varepsilon}_{\beta_2}+(\alpha_1+\alpha_2)\boldsymbol{\varepsilon}_{\alpha_1,\alpha_2}]$$

from which relation the single strains are obtained. The strains of the elements working in series are

$$\boldsymbol{\varepsilon}_{\beta_1} = \mathbf{K}_1^{-1}\boldsymbol{\sigma}$$

$$\boldsymbol{\varepsilon}_{\beta_2} = \mathbf{K}_2^{-1}\boldsymbol{\sigma}$$

whereas the (common) strains of the parallel assembly become

$$\boldsymbol{\varepsilon}_{\alpha_1,\alpha_2} = (\alpha_1+\alpha_2)\,(\alpha_1\mathbf{K}_1+\alpha_2\mathbf{K}_2)^{-1}\boldsymbol{\sigma}. \quad (9)$$

The phase stresses will be given by

$$\begin{aligned}
\boldsymbol{\sigma}_{\beta_1} &= \boldsymbol{\sigma}\\
\sigma_{\beta_2} &= \boldsymbol{\sigma}\\
\boldsymbol{\sigma}_{\alpha_1} &= \mathbf{K}_1\boldsymbol{\varepsilon}_{\alpha_1,\alpha_2} = \mathbf{K}_1\,(\alpha_1+\alpha_2)\,(\alpha_1\mathbf{K}_1+\alpha_2\mathbf{K}_2)^{-1}\boldsymbol{\sigma}\\
\boldsymbol{\sigma}_{\alpha_2} &= \mathbf{K}_2\boldsymbol{\varepsilon}_{\alpha_1,\alpha_2} = \mathbf{K}_2(\alpha_1+\alpha_2)\,(\alpha_1\mathbf{K}_1+\alpha_2\mathbf{K}_2)^{-1}\boldsymbol{\sigma}.
\end{aligned} \quad (10)$$

Note again that, if the stiffnesses coincide, the above formulae render a constant value of strain and stress for each element.

The above simple model can be enriched by adding more elements working in series and in parallel, with a certain degree of arbitrariness on the geometry of the assembly. Now the crucial question arises; which one is the correct, or more accurate, representative geometry? The precise answer to that question will not be given here, since the present work is still preliminary and more analysis needs to be done. Nonetheless, one can see that the answer depends on the characteristics of the material and on the ability one has to capture them with a mechanical analogy. For example, a binary composite with the same constituent phases may be related to slightly different ladder models if the size distribution of the dispersed phase varies; also, the shape of the dispersed particles may be of importance. All these observations lead one to conclude that the potentialities of the model can be used to their full extent only when the different factors characterizing its definition are taken into account.

Finally, a question arises about the importance of going through all the trouble of defining the model geometry if a simple constitutive matrix, the overall stiffness, is obtained at the end. The answer lies in the distinctive feature of the ladder model, namely that each single component is not required to follow a linearly elastic behavior, hence to possess a constant stiffness. Each building element may be subject to plastic flow or to any other sort of nonlinearity in the mechanical response, which will influence the overall behavior of the composite. The current constitutive matrix of a single element may be evaluated by following its stress and strain histories, which depend on the overall loading history, on the geometry of the model and on the mechanical behavior of the rest of the assembly. Obviously, elements of the ladder model formally representing the same phase may be characterized by different responses if their respective mechanical histories do not coincide. Considering again the model of Fig. 1(b), the distinction between phase 1 and phase 2 needs to be complemented by a further differentiation based on the different loading histories of the single elements. The stiffness matrix of eqn (7) should then be rewritten stressing that the constitutive behavior of the different components may differ even if they belong to the same phase, hence

$$\mathbf{K}_{\text{overall}} = (\beta_1+\beta_2+\alpha_1+\alpha_2)\,[\beta_1(\mathbf{K}_{\beta_1})^{-1}+\beta_2(\mathbf{K}_{\beta_2})^{-1}+(\alpha_1+\alpha_2)^2\cdot(\alpha_1\mathbf{K}_{\alpha_1}+\alpha_2\mathbf{K}_{\alpha_2})^{-1}]^{-1} \tag{11}$$

where the added subscripts denote further differentiation. A model similar to the one of Fig. 1(b) will be employed in the next section to describe a material possessing some of the constitutive features of concrete. In that case, in order to take into account the loading/unloading conditions for each component, the elastic stiffness will be used to compute the initial strains and the actual loading history will be determined by a modified Newton–Raphson iteration.

It is now time to comment about the reason for the designation of the new model as the "ladder" model. The main reason is historical, since some relaxation models which have been proposed for the mechanical interpretation of viscoelastic behavior were made of elastic and viscous elements assembled in a fashion similar to the geometries that have been discussed. In addition, consider the possibility of further differentiation and splitting of each constitutive element into an assembly of smaller ones; since the representative volume fractions decrease, one can represent or capture all the length scales of the mechanical response. Elements belonging to the same phase may be conceived as having somewhat different constitutive behavior if their representative length scales are very different. Hence, the mechanical analogy could possibly allow the representation of different constitutive behaviors at diverse micromechanics levels or, in other words, to go down the ladder of the microscale characteristic lengths; this idea constitutes the other reason behind the designation "ladder method."

3. AN APPLICATION TO A CONCRETE-LIKE MATERIAL

As already mentioned, concrete is a binary composite characterized by a random distribution of the two phases and by a rather wide range of microstructure dimensions. In this section constitutive laws will be presented which may be used to study the behavior of concrete. In the next section the ladder method will be applied to a binary composite in which the two phases possess some of the constitutive features of the constituent phases of this material, mortar and aggregate. Needless to say, there are many other characteristics which this simple numerical experiment neglects, such as the non-associated plastic flow and the hysteretic behavior under cyclic loading; these aspects will be considered in future studies.

Since most of the inelastic deformation of concrete is located in the matrix phase (mortar) or at the interface between mortar and aggregate, a particular effort will now be devoted to the definition of their constitutive laws, while the dispersed phase (aggregate) will not receive a great deal of attention. In particular, in the numerical studies presented in the next section, the constitutive behavior of the dispersed phase will be considered linearly elastic and isotropic. A detailed explanation of a distributed damage model, which may be used to describe the softening behavior of mortar, will now be introduced, followed by the definition of a compact failure criterion for the interface.

3.1. *Constitutive model for the matrix phase*

The constitutive model for mortar intends to take into account the effects on the material response originated by the onset and development of microcracks. This task is the proper objective of the field of damage mechanics, which tries to define phenomenological quantities (damage variables) governing the nonlinear behavior of the material under arbitrary loading histories. If one looks at the current literature, one sees that there are several, sometimes conflicting, definitions of the damage variable. A large part of the current studies assumes that the inelastic behavior due to microcracking can be expressed by a single, scalar damage parameter [Kachanov (1958); Lemaitre and Chaboche (1978, 1985); Lemaitre (1985, 1987); for a more thorough list, see the review paper by Krajcinovic (1989)]. Unavoidably, the definition of damage as a scalar conflicts with the real, anisotropic material behavior occurring when the manifestation of damage is a distribution of oriented cracks and not an ensemble of spherical voids. Some authors [for example, Ortiz (1985); Simo and Wu (1987a)] try to accommodate for the anisotropy in the material response by expressing the change in compliance as a function of damage and of the current stresses. If one considers a few selected loading histories, one can observe that the predictions of these models are physically reasonable only for the case of proportional loading.† A step above these first order approximations one finds the studies of damage conceived as a second rank tensor (Kachanov, 1980; Murakami, 1988; Dragon and Mroz, 1979), which are mainly characterized by the effort of describing damage through an integral over the distribution of size and of orientation of the microcracks. Finally, one finds the fourth and eighth order tensors as possible candidates for the description of damage (Horii and Nemat-Nasser, 1983; Hashin, 1986; Chaboche, 1978, 1979). All of these efforts are characterized by the non-trivial problem of relating a physical, possibly measurable quantity such as the extension and orientation of microcracks to the change in material response; the connection is not easy to establish if one enters the nonlinear regime where cracks start to interact and coalesce.

In the present work, as suggested by Ortiz (1985), the proposed damage variable will not be defined by a more or less directly measurable physical quantity, for example the area of the microcracks or the volume of the pores in the material, but instead will be expressed by the change in the elastic properties of the solid. This approach has the advantage of bypassing the development of a law relating the physical manifestations of damage and the change in constitutive behavior, which law would need to take into account factors such as the interaction between microcracks, percolation of the defects and other nonlinearities which make a clean theoretical derivation almost impossible. Instead, the damage evolution law is obtained indirectly but unequivocally from experimental data giving the behavior of the damaged material.

Following these considerations, the proposed damage model assumes that damage can be described by a symmetric tensor $\boldsymbol{\mu}$ so that the compliance of the material can be written, in component form, as

$$C_{ijkl} = C^0_{ijkl} + \Delta C^{\text{damage}}_{ijkl} = C^0_{ijkl} + \mu_{ik}\mu_{jl} \tag{12}$$

where C^0_{ijkl} represents the compliance of the undamaged solid and $\Delta C_{ijkl} = \mu_{ik}\mu_{jl}$ represent the corresponding increase due to microcracking. From purely physical considerations, it is obvious that the damage tensor $\boldsymbol{\mu}$ cannot have negative principal values. Also, Eqn (12) is only valid for positive applied stresses; if some principal stress is negative (compressive), then the added compliance due to microcracking should not appear in the corresponding direction, hence the equation will require some modifications which will be detailed later in this section.

As already mentioned, the above eqn (12) is actually a definition of the damage variable, which becomes a second order tensor. It is worth noticing that eqn (12) represents a rather abrupt change from current damage models which tend to express damage as a

† The reason for the incongruous predicted behavior comes from the choice of a (scalar) critical stress controlling the evolution of damage, independently from the preferred orientation of the microcracks.

function of a dyadic product of microcrack orientation, integrated over the distribution of microcracks. The above increase in compliance does not follow from the definition of a dyadic product between some vectors or tensors. Comparing the current one to previous phenomenological models, one can see that the order of indices differs from the usual notation for the direction of damage (Ortiz, 1985; Simo and Wu, 1987a); the current law in fact removes some of the inconsistencies which are encountered when non-proportional loading histories are considered.† With the current model, referring to a principal frame for the symmetric tensor $\boldsymbol{\mu}$, one can see instead that all the additional terms of the form ΔC_{iill} (no sum on i and l, i different from l), responsible for those inconsistencies, vanish because they are given by the product $\mu_{il}\mu_{il}$, also vanishing in that reference frame.

One can also show that the additional compliance due to damage is characterized by minor and major symmetries. The major symmetry is a direct consequence of the symmetry of the damage tensor $\boldsymbol{\mu}$, whereas the minor ones are due to the fact that at least two of the indices i, j, k, l (k and i in the following example) need to coincide, since they range from one to three; hence any component ΔC_{ijkl} has to be of the form

$$\Delta C_{ijkl} = \mu_{ij}\mu_{il} \quad \text{or} \quad \Delta C_{ijkl} = \mu_{ii}\mu_{jl} \quad (\text{no sum}, i = k). \tag{13}$$

It is a matter of evidence to see that the remaining two indices j and l are perfectly interchangeable.

Equation (12) is potentially applicable to any kind of material undergoing mechanical damage, but attention shall now be restricted to damage growth rate laws which seem to be appropriate for describing the inelastic behavior of the matrix phase of concrete, mortar. Following Ortiz (1985), a critical stress variable governing the onset and growth of damage needs to be defined; its definition will contain a tensor valued function of the damage components and also of the material elastic parameters. As detailed in a later section where the yield condition will be addressed, the critical stress will define an upper (tensile) bound to the stress states which the material can sustain compatibly with the current amount of damage.

In the uniaxial case, Ortiz (1985) used Smith and Young's (1955) law to represent the stress–strain relation. This law reads

$$\sigma = E_0\,\varepsilon \exp\left(-\frac{\varepsilon}{e_t}\right) \tag{14}$$

where ε and σ represent the axial strain and stress, respectively, and e_t is the deformation at peak load. For a definition of the three-dimensional critical stress law one needs to extend the uniaxial relation to multiaxial loading conditions. Such an extension has to retain the physical features of Smith and Young's law, namely that softening is maximum perpendicularly to the plane with a larger projected area of the microcracks but also has to accommodate the simultaneous presence of damage along different directions. In such a respect, the law which will be proposed can be labelled linear, in the sense that the deriving yield parameter will be expressed in terms of the linear superposition of all past damage growth. The proposed three-dimensional linear extension of Smith and Young's law reads

$$\sigma_{ij} = \left(\frac{1}{E_0}\,\delta_{ip}\delta_{jq}\right)^{-1} \mathrm{EXP}\left(-\frac{\sqrt{\varepsilon_{pk}\varepsilon_{ql}}}{e_t}\right)\varepsilon_{kl} \tag{15}$$

where E_0 is the Young's modulus of the undamaged material. The exponential tensor EXP is defined as the tensor which, in a principal reference frame for any tensor $\mathbf{A}$, has principal values corresponding to the exponential of the principal values of $\mathbf{A}$. Furthermore, the quantity $\sqrt{\varepsilon_{pk}\varepsilon_{ql}}$, defined only for positive values of the principal strains which correspond

† For example, considering Ortiz's theory, a material which is loaded along two perpendicular axes and then unloaded, upon further loading on one axis only may have a lateral expansion due to damage which exceeds the elongation along the loading direction.

to tensile loading conditions, guarantees the symmetry of the new law with respect to the last two indices of the constitutive stiffness. It is straightforward to verify that the above relationship reduces to Smith and Young's law in the presence of uniaxial straining aligned with the reference axes.

Under tensile loading conditions, the critical stress can be obtained both from the extended Smith and Young's law as well as from the increase in compliance,

$$t_{ij} = \left(\frac{1}{E_0}\delta_{ik}\delta_{jl} + \mu_{ik}\mu_{jl}\right)^{-1} \varepsilon_{kl} \tag{16}$$

or

$$t_{ij} = \left(\frac{1}{E_0}\delta_{ip}\delta_{jq}\right)^{-1} \mathrm{EXP}\left(-\frac{\sqrt{\varepsilon_{pk}\varepsilon_{ql}}}{e_t}\right)\varepsilon_{kl} \tag{17}$$

where, using some licence of notation, $\left(\frac{1}{E_0}\delta_{ik}\delta_{jl} + \mu_{ik}\mu_{jl}\right)^{-1}$ is taken to represent the (i, j, k, l)th component of the effective stiffness for uniaxial loading. From these equations, one can observe that the following relation must hold:

$$\left(\frac{1}{E_0}\delta_{ik}\delta_{jl} + \mu_{ik}\mu_{jl}\right)^{-1} = \left(\frac{1}{E_0}\delta_{ip}\delta_{jq}\right)^{-1} \mathrm{EXP}\left(-\frac{\sqrt{\varepsilon_{pk}\varepsilon_{ql}}}{e_t}\right). \tag{18}$$

From the above equation, one then obtains

$$\mathrm{EXP}\left(\frac{-\sqrt{\varepsilon_{mk}\varepsilon_{nl}}}{e_t}\right) = \frac{1}{E_0}\delta_{mi}\delta_{nj}\left(\frac{1}{E_0}\delta_{ik}\delta_{jl} + \mu_{ik}\mu_{jl}\right)^{-1}. \tag{19}$$

Setting $k = n$ and summing gives

$$\mathrm{EXP}\left(-\frac{\varepsilon_{ml}}{e_t}\right) = \frac{1}{E_0}\delta_{mi}\delta_{nj}\left(\frac{1}{E_0}\delta_{in}\delta_{jl} + \mu_{in}\mu_{jl}\right)^{-1} \tag{20}$$

or, upon inversion of the exponential,

$$\varepsilon_{ml} = \varepsilon_t \,\mathrm{LOG}\left(\frac{1}{E_0}\delta_{ni}\delta_{lj}\right)^{-1}\left(\frac{1}{E_0}\delta_{im}\delta_{jn} + \mu_{im}\mu_{jn}\right). \tag{21}$$

From that expression and from the expression of the critical stress eqn (16), one obtains

$$t_{ij} = E_0(\delta_{ik}\delta_{jl} + E_0\mu_{ik}\mu_{jl})^{-1}\,\mathrm{LOG}\,(\delta_{kl} + E_0\mu_{ln}\mu_{nk})e_t. \tag{22}$$

It is a matter of some algebra, which can be simplified by writing eqn (22) in a principal reference frame for $\boldsymbol{\mu}$, to verify that the principal values of the critical stress tensor are

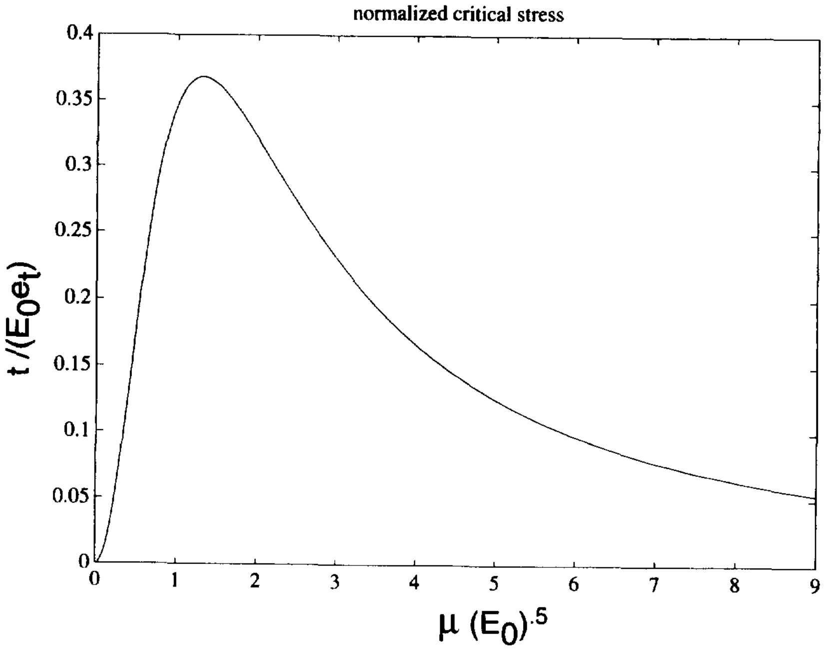

Fig. 2. Critical stress as a function of damage for a uniaxially damaged material.

always non-negative. In the case of damage growing along one direction only, say μ_{11}, the behavior of the corresponding critical stress is shown in Fig. 2. For sake of notation, it is convenient to express the above relation in tensorial form; if one denotes by $\mathcal{M}$ the dimensionless tensor of components $\mathcal{M}_{ijkl} = \delta_{ik}\delta_{jl} + E_0\mu_{ik}\mu_{jl}$, and by ":" the double inner product of two tensors $((\mathbf{A}:\mathbf{B})_{ijkl} = A_{ijpq}B_{pqkl})$, then the expression for the critical stress can be written as

$$\mathbf{t} = E_0 e_t \mathcal{M}^{-1} : \mathrm{LOG}\,(\mathcal{M} : \mathbf{I}), \tag{23}$$

$\mathbf{I}$ being the identity tensor of rank two.

The above equation is similar to that derived by Ortiz (1985) in the framework of his scalar damage model for concrete. The differences between the current and Ortiz's theory are due to the different definition of damage variable, represented by a tensor in the present work. As in the properly uni-dimensional case (Ortiz, 1985), one can verify that the critical stress of the three-dimensional model vanishes if the material is undamaged ($\boldsymbol{\mu} = 0$), hence the damaging process starts as soon as the material is loaded.

Now the condition of some negative applied stress will be considered. In order to do so, it is convenient to introduce the concept of positive projection of a tensor. As addressed more extensively in the Appendix, we define the projection of the damage tensor $\boldsymbol{\mu}$ along the positive directions of the applied stress $\boldsymbol{\sigma}$ as the tensor of components (Simo and Wu, 1987a)

$$\mu_{ij}^{+(\sigma)} = \left[\sum_{p:\sigma_p>0} n_i^{(p)} n_k^{(p)}\right]\left[\sum_{p:\sigma_p>0} n_j^{(p)} n_l^{(p)}\right]\mu_{kl} \tag{24}$$

where $n_i^{(p)}$ represents the ith component of the eigenvector corresponding to the positive principal stress σ_p, the summations being extended only to the positive principal directions of $\boldsymbol{\sigma}$. Then the law giving the added compliance due to microcracking is still valid for a general stress state if one replaces the damage tensor by its projections along the positive directions of the applied stress.

3.1.1. *Yield condition.* For a material which undergoes damage isotropically, the damage rate is usually obtained considering the variation in the yield function and requiring it to vanish under loading conditions. This procedure is made possible by the isotropic dependence of the change in yield function on the stress increment. In the case of anisotropic damage, this requirement becomes difficult to impose because the yield function does not depend only on the stress state but also on its orientation with respect to the material itself and its past history. The yield criterion is therefore associated with an entity which distinguishes the directions along which a stress increase is incompatible with a constant critical stress or damage from those where the threshold is still far away from the current loads. A suitable approach could require, for example, that the difference between the applied stress and the critical stress does not have positive eigenvalues (the principal values of the critical stress, as shown earlier, cannot be negative). Graphically this requirement can be visualized using Cauchy's stress quadric; for a generic symmetric tensor $\mathbf{A}$ with positive eigenvalues, the Cauchy surface in three-dimensional space can be written in the form

$$A_{ij}x_i x_j = k_0 > 0 \tag{25}$$

where k_0 is an arbitrary positive constant. If one sets $\mathbf{A} = \boldsymbol{\sigma}$ (applied stress), then the normal stress σ_n acting on any material surface with normal $\boldsymbol{\nu}$ is related to the corresponding radius of the Cauchy surface in the form

$$\sigma_n = \frac{k_0}{x_i x_i} \tag{26}$$

where $x_i x_i$ represents the squared distance between the origin and the point on the Cauchy surface aligned with $\boldsymbol{\nu}$, i.e.

$$\nu_i = \frac{x_i}{x_k x_k}. \tag{27}$$

The yield condition can then be obtained by imposing that the applied stress quadric always remain external to the critical stress quadric for any direction $\boldsymbol{\nu}$ considered. Evidently, if one or more eigenvalues of the applied stress quadric are negative, the above eqn (25) may have no real solution for some domain of directions $\boldsymbol{\nu}$, so the yield condition is always satisfied in that domain.

How does the anisotropic yield condition affect the stress increments? If a stress change modifies the stress state so that the applied stress quadric always remains outside the critical one, no increment in damage is necessary. Conversely, if the two surfaces are already in contact at some point, then the yield condition along that particular direction has to be taken into account. A method will now be shown which is suitable for incorporating the directionality of the yield condition and which relies on the use of the projection operator, as explained in the Appendix. This operator will be based on the critical stress difference $\mathbf{z} = \boldsymbol{\sigma} - \mathbf{t}$, namely the difference between the applied and the critical stress.

Starting from the critical stress difference $\mathbf{z}$, one can construct a step function tensor $\mathbf{H}(\mathbf{z}) = \mathbf{H}^{+(\mathbf{z})}$ which, in the principal reference frame for $\mathbf{z}$, assumes a diagonal form with zero elements if the corresponding elements of $\mathbf{z}$ are negative, and with unitary values if the corresponding elements of $\mathbf{z}$ are positive or vanish. Through the step function tensor, one can also define the four-dimensional projection operator $\mathbf{P}^{+(\mathbf{z})}$ given, in component form, as

$$P_{ijkl}^{+(\mathbf{z})} = H_{ik}^{+(\mathbf{z})} H_{jl}^{+(\mathbf{z})}. \tag{28}$$

The rate form of the yield condition can then be written by weighting the elastic change of the applied stress through the projection operator and requiring that the tensor†

$$\dot{Y}_{ij} = P_{ijpq}^{+(\mathbf{z})} \frac{\partial \sigma_{pq}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \frac{\partial \sigma_{ij}}{\partial \mu_{kl}} \dot{\mu}_{kl} - \dot{t}_{ij}, \tag{29}$$

has no positive eigenvalues, subject to the condition that all eigenvalues of $\dot{\boldsymbol{\mu}}$ be non-negative. The effect of the projection operator in the above equation is to remove from consideration all the elastic stress increments which correspond to directions where the corresponding normal tractions are still far away from the bounding surface defined by the critical stress tensor. As mentioned in the Appendix, the definition of the projection operator based on the Heaviside step function leads to numerical difficulties and is probably not too realistic. It therefore seems justifiable to replace the kernel functions in the projection operator with some function going smoothly from zero to one as its argument becomes positive. An ideal candidate is the projection operator based on the error function (see the Appendix),

$$P_{ijkl}^{+(\mathbf{z})} = \mathrm{ERF}_{ik}^{(\mathbf{z})}\, \mathrm{ERF}_{jl}^{(\mathbf{z})} \tag{30}$$

where

$$\mathrm{ERFC}\,(\mathbf{z}) = \sum_{\alpha=1}^{3} \mathrm{erfc}\,(\lambda_\alpha, \sigma_v) n_i^\alpha n_j^\alpha \tag{31}$$

σ_v being the variance of the distribution and λ_α, $\alpha = 1, 2, 3$ the set of real eigenvalues of the critical stress difference $\mathbf{z} = \boldsymbol{\sigma} - \mathbf{t}$.

For the very special case of loading in all directions, one could solve directly for the damage increment in the form

$$\dot{\mu}_{kl} = \left\{ \frac{\partial t_{ij}}{\partial \mu_{kl}} - \left[\frac{\partial \sigma_{ij}}{\partial \mu_{kl}} \right] \right\}^{-1} P_{ijpq} \left[\frac{\partial \sigma_{pq}}{\partial \varepsilon_{rs}} \dot{\varepsilon}_{rs} \right] \tag{32}$$

but generally this condition cannot be verified *a priori*. In fact, the solution to eqn (29) may not even be unique unless one imposes some additional conditions, as will be done in the next section for a case where the principal axes of damage do not rotate. The additional condition in that case will be chosen as the maximization of the total damage increment, sum of three damage increments along three known perpendicular directions; the yield condition can then be implemented using the tools of linear programming.

The partial derivatives appearing in eqn (32) can be obtained by noting that, given any non-singular tensor $\mathbf{A}$, the time derivative of its inverse is given by

$$\frac{\partial \mathbf{A}^{-1}}{\partial t} = \dot{\mathbf{A}}^{-1} = -\mathbf{A}^{-1} \dot{\mathbf{A}} \mathbf{A}^{-1} \tag{33}$$

hence

† The approximation of small deformations and small material rotations is adopted throughout this study. The time derivatives of all tensorial quantities are therefore computed by simple time differentiation of their components, without considering corotational derivatives as, for example, Jaumann's derivative.

$$\frac{\partial \sigma_{ij}}{\partial \mu_{kl}} = -[C_{ijpq}+\mu_{ip}\mu_{jq}]^{-1}(\mu_{pm}\delta_{qk}\delta_{nl}+\mu_{qn}\delta_{pk}\delta_{ml})[C_{mnrs}+\mu_{mr}\mu_{ns}]^{-1}\varepsilon_{rs} \tag{34}$$

or, after some rearranging which takes advantage of the minor symmetries of the total compliance tensor,

$$\frac{\partial \sigma_{ij}}{\partial \mu_{kl}} = -2[C_{ijpk}+\mu_{ip}\mu_{jk}]^{-1}[C_{mlrs}+\mu_{mr}\mu_{ls}]^{-1}\mu_{pm}\varepsilon_{rs}. \tag{35}$$

If one approximates† the derivative of the logarithmic tensor of a symmetric matrix **A** as

$$\frac{\mathrm{dLOG}\,(\mathbf{A})}{\mathrm{d}t} \sim \mathrm{symm}\,(\dot{\mathbf{A}}\mathbf{A}^{-1}) = \tfrac{1}{2}(\dot{\mathbf{A}}\mathbf{A}^{-1}+\mathbf{A}^{-1}\dot{\mathbf{A}}), \tag{36}$$

one can obtain the derivative of the critical stress with respect to the damage increment. In compact form this derivative reads

$$\nabla\boldsymbol{\mu}\mathbf{t} = E_0 e_t\,[-\mathcal{M}^{-1}:\nabla\boldsymbol{\mu}(\mathcal{M}):\mathcal{M}^{-1}+\tfrac{1}{2}\mathcal{M}^{-1}:(\nabla\boldsymbol{\mu}(\mathcal{M}):\mathcal{M}^{-1}+\mathcal{M}^{-1}:\nabla\boldsymbol{\mu}(\mathcal{M})):\mathbf{I}] \tag{37}$$

where $\nabla\boldsymbol{\mu}\mathbf{A}$ denotes the derivative of the tensor **A** with respect to the damage variable $\boldsymbol{\mu}$ whereas in component form,

$$(\nabla\boldsymbol{\mu}\mathbf{A})_{ijkl} = \frac{\partial A_{ij}}{\partial \mu_{kl}}. \tag{38}$$

3.2. *An additional failure mode of the ladder representation*

The representation of the real composite material by the conventions of the ladder model allows some considerations about the non-uniform states of stress arising throughout the solid and their consequences. These further observations induce possible explanations for the failure modes of these materials under compression, modes which are characterized by the onset of microcracks parallel to the loading axis preceding the total disgregation of the solid. Poisson effects (i.e. different lateral expansion or contraction for two phases under uniaxial loading which causes a tensile and a compressive lateral stress) have been usually proposed to be the cause of these typical failure surfaces. While these effects are still retained in the local behavior of the elements composing the ladder model, an additional failure mechanism will now be addressed.

Consider the example of Fig. 3 where an idealized situation of a geometrically regular material microstructure is shown next to its ladder model representation; such an idealized material is supposed to be subjected to some overall compressive stress. In the framework of the ladder model, one can compute and analyse only uniform stress states located at idealized junctions between the different representative elements. These idealized stress states would correspond to an averaging of the non-uniform spatial distributions of the real physical world.

When two elements working in parallel are subjected to two quite different states of stress, the material which surrounds them (or, in the ladder abstraction, works in series with them) has to redistribute a more or less uniform stress state received at one end into a rather uneven condition at the interface with the two elements in parallel. One can see that this redistribution corresponds to a stress increase in the interior of the representative volume, a stress increase which cannot be captured directly by the ladder model since this model only deals with the stresses at the element interfaces/junctions. This additional

† The problem of differentiating a tensor valued function of a tensor was approached by Hoger (1986), who actually gave a closed form solution for the derivative of the logarithm of a tensor. Nonetheless, considering the complexity of Hoger's solution, an approximate but simpler relationship was adopted for this derivative with an error of the order of a few percent if the damage increments are kept sufficiently small; the approximate relation becomes exact if the principal axes of damage do not rotate.

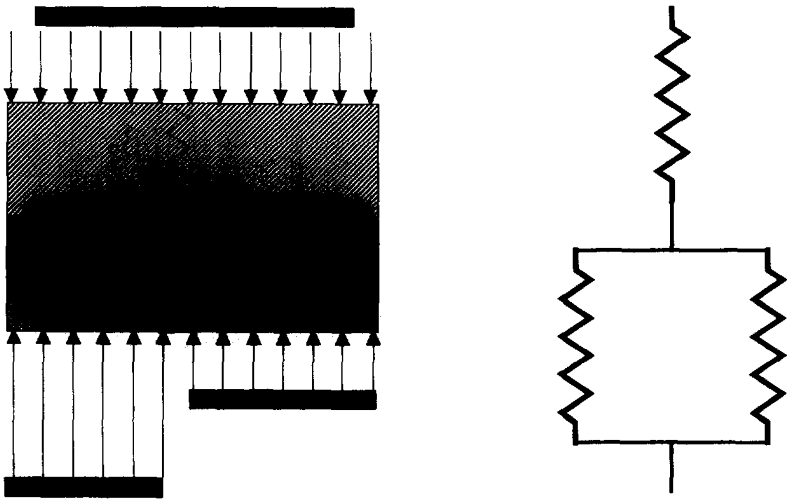

Fig. 3. An idealized geometry of a composite and the corresponding ladder model representation.

loading has to be considered when failure modes are addressed. In the idealized geometry being considered, the stress difference between the two elements working in parallel is likely to induce a stress concentration, hence progressive degradation and failure, at the boundary where the three elements intersect.

A question now arises on how to take into account this mechanism of degradation in the overall response of the ladder model. Some considerations can help in defining those criteria.

- As opposed to the damage model previously defined for the softening phase, where the material did not present any directionality with respect to the loading axis, now there is a boundary between ladder model elements working in parallel, and the response has to be averaged over all possible orientations of this boundary.
- For the above reason, an anisotropic damage condition with regard to the failure of these elements would be inconsistent, being subjected to a preceding process of averaging. A scalar model of degradation must be employed.
- In the spirit of the ladder model, one may assume that the whole assembly (one element in series with two elements in parallel) is representative of a statistical population of volumes of material. Over this statistical population, one may define an expected value of some parameter, based on the stress difference in the parallel assembly, that governs the failure of these representative volume elements.
- If one assumes that the statistical distribution of the failure parameter has a certain shape (e.g. Gaussian), then the fraction of elements which have failed can be easily computed from the current value of the critical variable. Again in the spirit of the ladder model, the response of the assembly will be proportional to the remaining fraction of load carrying members.

The critical scalar parameter has to be defined through some function of the invariants of the stress difference, i.e. the difference between the stress states of two elements working in parallel. Since the sign of the stress difference is arbitrary, a good candidate for the critical function cannot distinguish between tensile and compressive stresses. For the sake of simplicity, in this work the square root of the second invariant of the parallel stress difference $I_2(\Delta\boldsymbol{\sigma})$ will be considered as the critical variable. Another possible choice, not considered here, would be the second invariant of the deviatoric stress difference. It will also be assumed that the distribution of microgeometries and physical bonds is such that the statistical distribution of failed elements is represented by a Gaussian curve centered at the reference value of the critical variable $\sqrt{I_2^{\text{crit}}} = s_0$, and having a certain variance s_v. The actual stress carried by an assembly of elements $\boldsymbol{\sigma}_{\text{effective}}$ is then obtained by multiplying the initially computed stress, in the absence of this failure mechanism $\boldsymbol{\sigma}$, by the fraction of

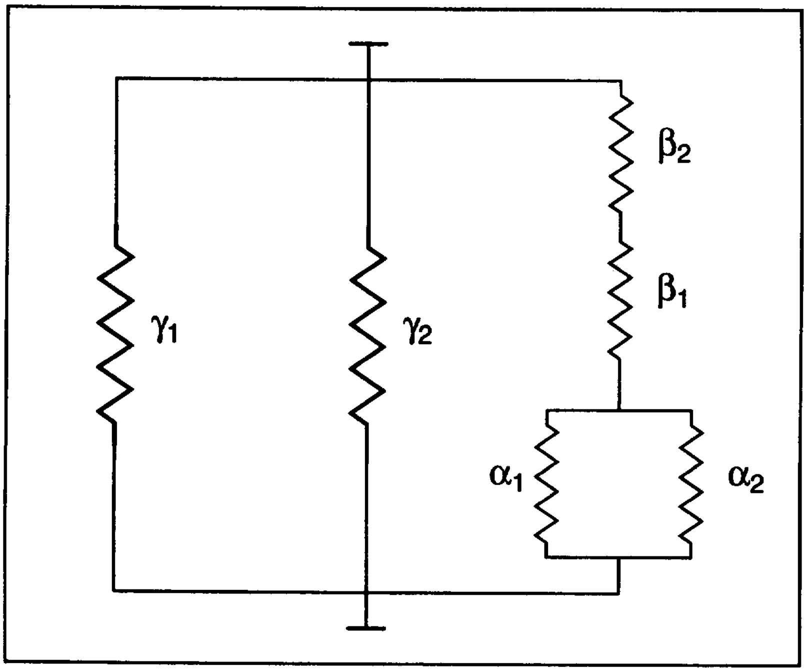

Fig. 4. A ladder model representation of the constitutive behavior of a concrete-like material.

elements which are still carrying load; this fraction can be written in terms of the complementary error function of the stress overshoot,

$$\sigma_{\text{effective}} = \sigma \operatorname{cerf}(\Delta\sigma, s_0, s_v) = \sigma\left[1-\operatorname{erf}\left(\frac{\sqrt{I_2(\Delta\sigma)}-s_0}{s_v}\right)\right]. \quad (39)$$

In the numerical results which will be now presented, the above relation (39) will be denoted as the "interface" correction to the stress state. This definition is motivated by the presence of an abstract connection (model interface) between two parallel elements in the ladder model and also by the fact that the stress redistribution between these two elements can locally affect the physical boundary between them (real interface).

4. THE NUMERICAL STUDY OF A CONCRETE-LIKE MATERIAL

4.1. *The ladder model*

The ladder model used to represent the constitutive behavior of the binary composite is shown in Fig. 4, where the addition of two elements working in parallel, having volume fractions γ_1 and γ_2, brings the effective stiffness matrix of eqn (11) to the expression

$$\mathbf{K}_{\text{overall}} = \frac{1}{\gamma_1+\gamma_2+\beta_1+\beta_2+\alpha_1+\alpha_2}\{\gamma_1\mathbf{K}_{\gamma_1}+\gamma_2\mathbf{K}_{\gamma_2}+(\beta_1+\beta_2+\alpha_1+\alpha_2)^2$$
$$\cdot[\beta_1(\mathbf{K}_{\beta_1})^{-1}+\beta_2(\mathbf{K}_{\beta_2})^{-1}+(\alpha_1+\alpha_2)^2(\alpha_1\mathbf{K}_{\alpha_1}+\alpha_2\mathbf{K}_{\alpha_2})^{-1}]^{-1}\} \quad (40)$$

where the initial values taken by the constitutive matrices are those defining the elastic behavior of phase 1 or 2, depending upon their second indices. Obviously the above stiffness is only useful for the initial definition of the strains affecting each component of the ladder model, which strains are subsequently adjusted iteratively when the effective elastic and inelastic stresses are computed. All the volume fractions $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are set equal to 1/6; the subscript 1 indicates that the particular element belongs to the softening matrix phase and a subscript 2 denotes an element following the constitutive behavior of aggregates.

For clarity reasons in the plots being presented, the different elements will be labelled with their corresponding volume fraction.

4.2. *Results and discussion*

The results of the mixture theory and of the ladder model will now be presented for a binary composite subjected to a prescribed strain history along a given axis. All considerations regarding the influence of the proposed constitutive law on mechanical instabilities and related issues [Hill (1958), (1959), (1961); also recently Losi (1994)] will not be addressed here and will be the subject of future work. One of the two phases is supposed to obey the anisotropic damage laws [eqns (12), (22) and (29)], while the other is taken to have a linearly elastic behavior. The initial Poisson's ratio for the softening phase is equal to 0.2, while the value for the linearly elastic phase equals 0.25. The following elastic parameters summarize the characterization of the material.

Phase 1 (mortar):

$$E_{0,1} = 5000\,\text{ksi} \quad G_{0,1} = 2083\,\text{ksi} \quad e_t = 0.4\times10^{-3} \quad \sigma_v = 0.1\,E_{0,1}e_t.$$

Phase 2 (aggregate):

$$E_{0,2} = 7500\,\text{ksi} \quad G_{0,2} = 3000\,\text{ksi}; \quad \text{linearly elastic.}$$

Maximum stress difference for elements working in parallel:

expected value of maximum stress difference: $s_0 = 4E_{0,1}e_t$

corresponding variance: $s_v = 4E_{0,1}e_t$.

Only cases of proportional straining will be presented. Under these conditions, with regard to the phase which undergoes damage induced softening, the expressions giving the critical stress and the yield condition can be greatly simplified since the principal axes of damage do not rotate and remain aligned with the loading axes. In the reference frame aligned with these axes, the critical stress can be written as

$$\mathbf{t} = E_0 e_t \begin{bmatrix} \dfrac{\log(1+E_0\mu_{11}^2)}{1+E_0\mu_{11}^2} & 0 & 0 \\ 0 & \dfrac{\log(1+E_0\mu_{22}^2)}{1+E_0\mu_{22}^2} & 0 \\ 0 & 0 & \dfrac{\log(1+E_0\mu_{33}^2)}{1+E_0\mu_{33}^2} \end{bmatrix}. \tag{41}$$

Analogously, the three strain components ε_{11}, ε_{22}, ε_{33} can be concisely written in terms of the corresponding stresses in the form

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{Bmatrix} = \begin{bmatrix} C_{1111}+\mu_{11}^2 & C_{1122} & C_{1133} \\ C_{2211} & C_{2222}+\mu_{22}^2 & C_{2233} \\ C_{3311} & C_{3322} & C_{3333}+\mu_{33}^2 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix} \tag{42}$$

which allows an easy computation of the partial derivatives $\partial\sigma_{ij}/\partial\mu_{rs}$. Under the requirement of positive damage growth ($\dot{\mu}_{ii} \geqslant 0$, no sum on i), the system of equations which defines the yield condition reduces to a linear programming problem. This problem cannot be solved consistently unless one imposes the solution to have some additional features. In the calculations presented here, the additional condition of maximization required that, for a given strain rate, the total damage rate $\dot{\mu}_{11}+\dot{\mu}_{22}+\dot{\mu}_{33}$ be maximum. The replacement of this

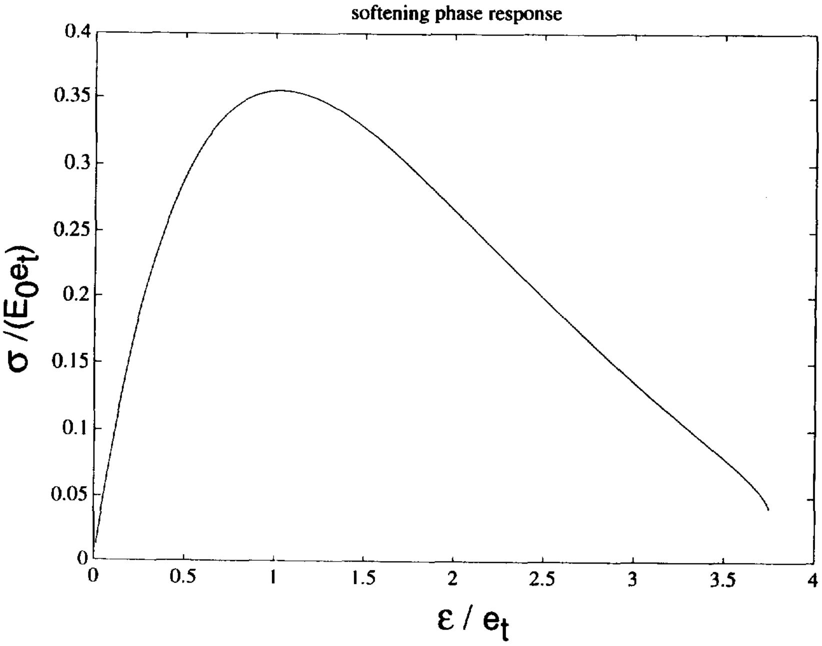

Fig. 5. Predictions of first order mixture theory; normalized stress/strain curve for the softening phase for a uniaxial tensile test, along the direction of loading.

extra condition with similar ones, such as the maximization of the inelastic strain increment, can be carried out with minimal effort. Following the above assumption, the three equations representing the yield conditions can be cast in the inequality form

$$\frac{\partial t_k}{\partial \mu_{kk}} \dot{\mu}_{kk} - \sum_i \frac{\partial \sigma_{kk}}{\partial \mu_{ii}} \dot{\mu}_{ii} \leqslant \sum_{p,q=1}^{3} Y_{pp}[C_{kkpq} + \mu_{kp}\mu_{kq}]^{-1} \dot{\varepsilon}_{pq} \geqslant 0 \quad k = 1,2,3 \tag{43}$$

where no sum on repeated indices is intended unless indicated. The above inequalities can be solved using linear programming techniques; the ones reported in Press *et al.* (1988) were used in this study.

Initially, the predictions of the first order mixture theory (only two phases with equal volume fractions) are presented. In the case of a uniaxial tensile test, Fig. 5 shows the stress history of the softening phase along the direction of loading. The only difference with respect to the critical stress curve of the isolated phase is a straighter portion of the curve in the descending part, possibly due to the smoothing effect of the projection operator appearing in the yield condition.

The behavior of the averaged mixture stress without the interface correction does not have any remarkable features, since every deviation from linearity is largely hidden by the elastic response of the second phase. However, if one applies the correction based on the second invariant of the stress difference, one obtains the curve shown in Fig. 6.

The results for the compression test with the interface correction are shown in Fig. 7. Along the axis of loading, both phases are in a compressive state whereas the lateral stress history of the softening phase shows tensile loading, as shown in Fig. 8. It is worth observing that, even within such a crude model, the absolute value of the maximum stress in compression of the composite is about four times higher then the corresponding one in tension, and all of this without recurring to "cross-effect" coefficients as it is sometimes done in the literature.

The results of the ladder model with six components shall now be considered. For the tension test with no interface correction, the more interesting plots are those giving the stress response of the two elastic components which are denoted in the model as having volume fractions β_2 and α_2. Due to the progressive softening of the matrix phase in series

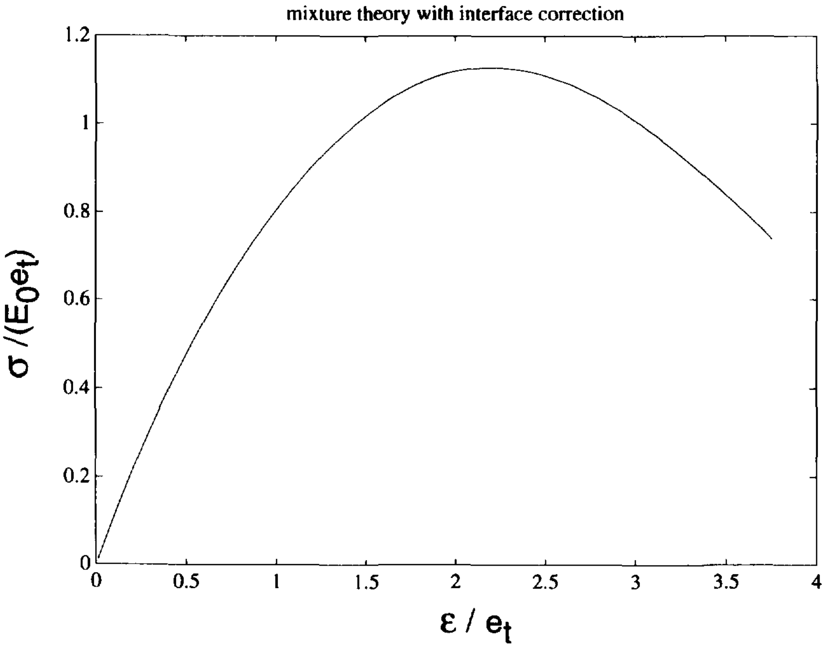

Fig. 6. Predictions of first order mixture theory with interface correction.

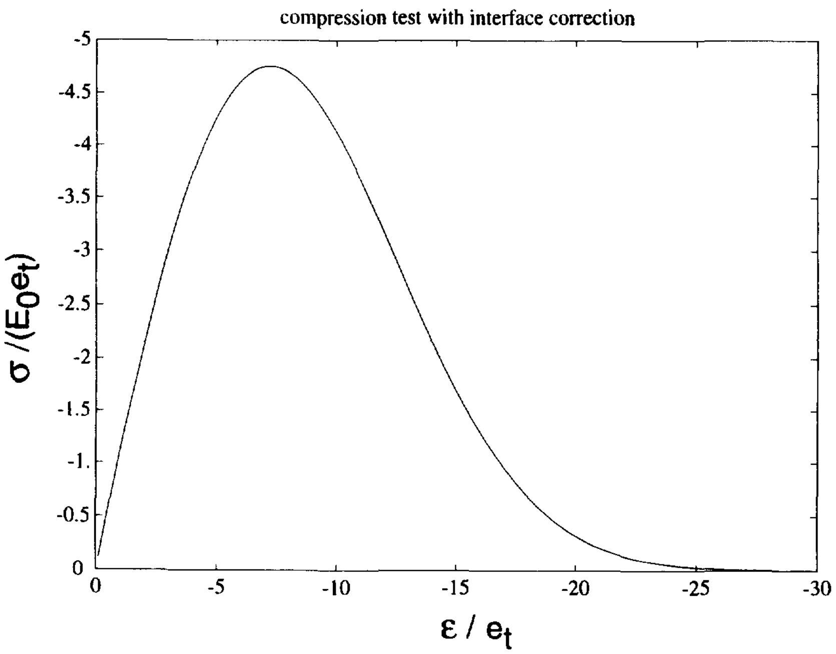

Fig. 7. Predictions of first order mixture theory with interface correction for the compression test.

with them, these two components see their applied stress, hence their total strain, diminish as the overall axial straining increases. These results are shown in Fig. 9.

The results of the compression test with no interface correction show no remarkable behavior since the presence of an elastic component working in parallel to everything else masks all the nonlinearities. The softening components are subjected laterally to tensile stress (hence, damage growth) as shown in Fig. 10.

If now one turns attention to the results where the interface correction is present, one can see that the ratio 1 : 4 of the strengths in tension and compression of the first order mixture model now become approximately 1 : 9. The interface correction was applied twice in the ladder model, first at the two components γ_1 and γ_2 in parallel, and then the resulting

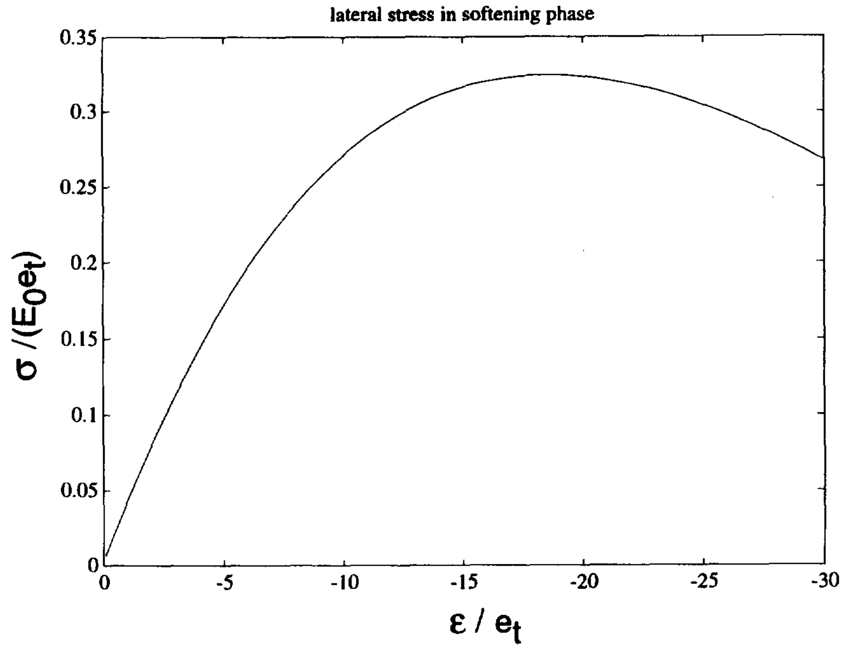

Fig. 8. Predictions of first order mixture theory. The plot shows the time history of the lateral stress in the softening phase.

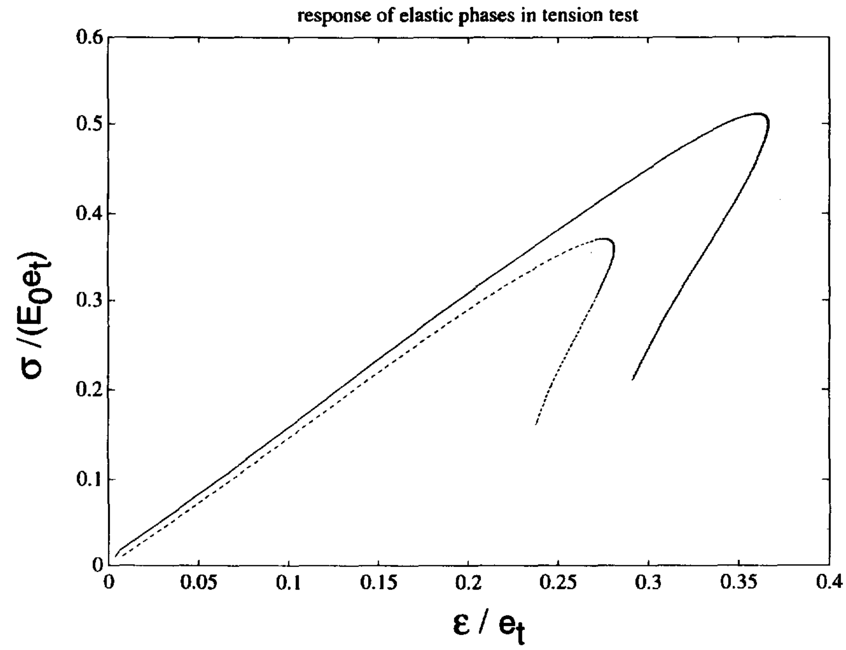

Fig. 9. Ladder model predictions; stress response of the elastic components participating in the series assembly. The continuous curve represents the loading history of the elastic element denoted in the model as α_2, whereas the dashed curve corresponds to the element β_2.

stress was compared with that of the series β_1, β_2, $\alpha_1+\alpha_2$. The results are shown in Figs 11 and 12.

5. CLOSURE

The intent of this work has not been to present definitive statements on how to represent the constitutive behavior of heterogeneous solids, but rather to sketch a new way

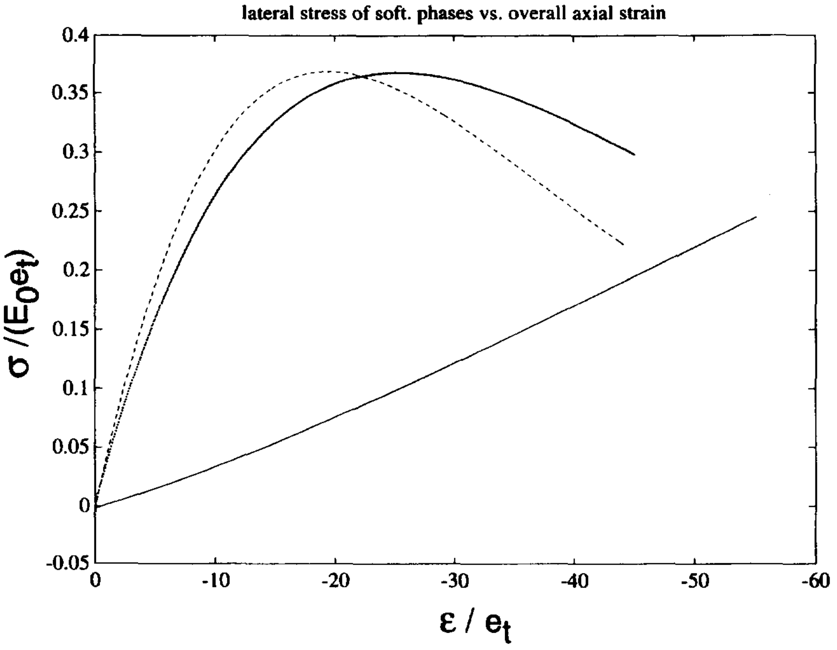

Fig. 10. Ladder model predictions; lateral stress of the softening components vs uniaxial compressive strain. The dashed line corresponds to the element α_1, the upper continuous line to the element γ_1 and the lower one to β_1.

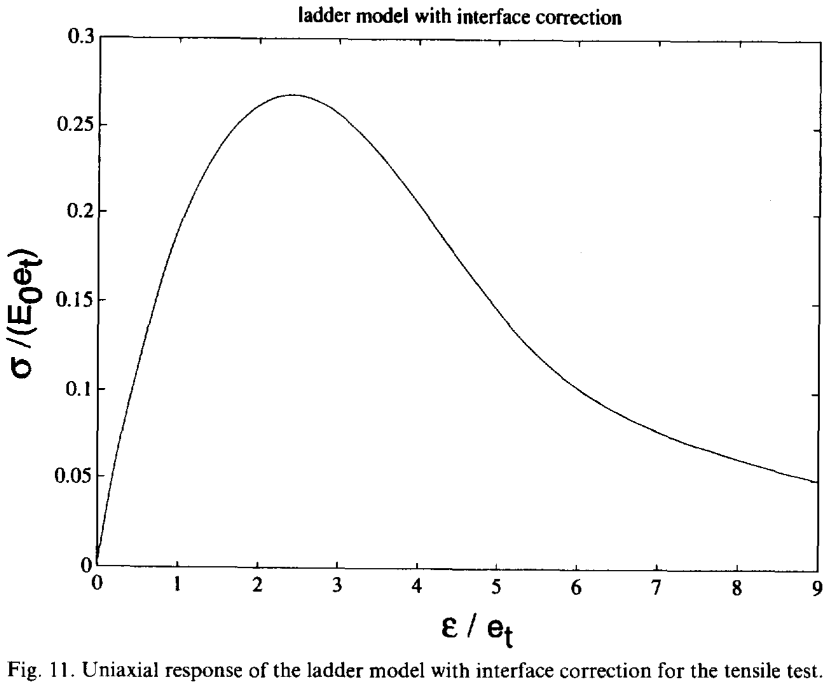

Fig. 11. Uniaxial response of the ladder model with interface correction for the tensile test.

of approaching the problem. Within these limitations, the presented results encourage further investigation in order to incorporate the many diverse aspects of the mechanical behavior of these materials. In addition, the proposed damage model incorporates three-dimensional and anisotropic features which certainly accompany the mechanical degradation of some of these composites, concrete remaining the primary example.

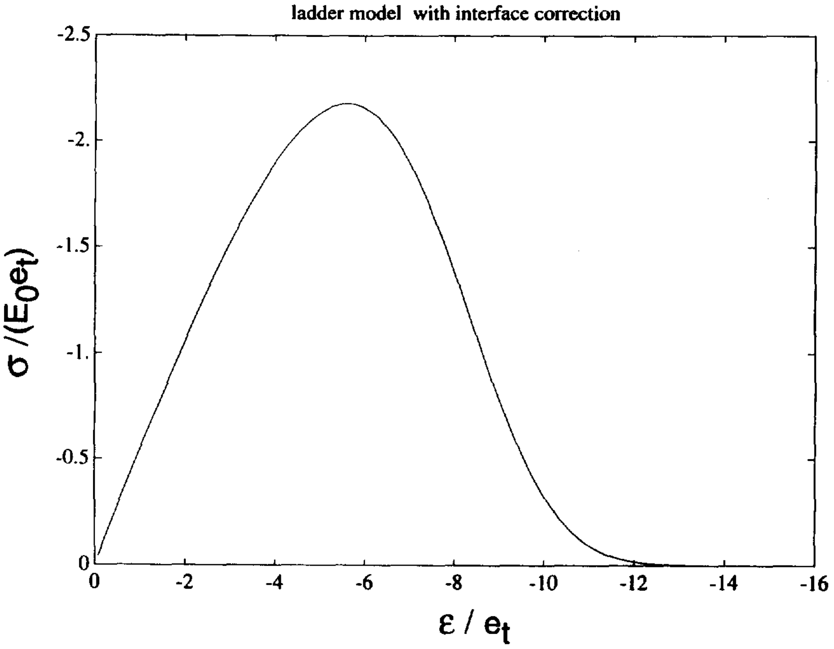

Fig. 12. Uniaxial response of the ladder model with interface correction for the compression test.

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REFERENCES

Bažant, Z. P. and Cedolin, L. (1991). *Stability of Structures.* Oxford University Press, New York.

Bažant, Z. P. and Oh, B. H. (1983). Crack band theory for fracture of concrete. *Mater. Construc.* **16,** 155–177.

Benveniste, Y. (1987). A new approach to the application of Mori–Tanaka's theory in composite materials. *Mech. Mater.* **6,** 147–157.

Chaboche, J.-L. (1978). Description thermodinamique et phenomenologique de la viscoplasticité cyclique. Office National d'Etudes at de Recherches Aerospatiales.

Chaboche, J.-L. (1979). Le concept de contrainte effective appliqué à l'elasticité et la viscoplasticité en presence d'un endommagement anisotrope. Report Colloq. Euromech 115, Grenoble, France.

Dragon, A. and Mroz, Z. (1979). A continuum model for plastic brittle behavior of rock and concrete. *Int. J. Engng Sci.* Vol. 17, p. 121–137.

Dvorak, G. J. and Benveniste, Y. (1992). On the thermomechanics of composites with imperfectly boned interfaces and damage. *Int. J. Solids Structures* **29**(23), 2907–2919.

Eshelby, J. D. (1957). The determination of the elastic field of an ellipsoidal inclusion and related problems. *Proc. R. Soc. London* **A241,** 376.

Green, A. E. and Naghdi, P. M. (1965). A dynamical theory of interacting continua. *Int. J. Engng Sci.* **3,** 231–241.

Hashin, Z. (1986). Analysis of stiffness reduction of cracked cross-ply laminates. *Engng Frac. Mech.* **25,** 771–778.

Hill, R. (1958). A general theory of the uniqueness and stability in elastic–plastic solids. *J. Mech. Phys. Solids* **6,** 236–249.

Hill, R. (1959). Some basic principles in the mechanics of solids without a natural time. *J. Mech. Phys. Solids* **7,** 209–225.

Hill, R. (1961). Uniqueness in general boundary value problems for elastic and inelastic solids. *J. Mech. Phys. Solids* **9,** 114–130.

Hilleborg, A., Modéer, M. and Petersson, P. E. (1976). Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement Concrete Res.* **6,** 773–782.

Hoger, A. (1986). The material time derivative of the logarithmic strain. *Int. J. Solids Structures* **22**(9), 1019–1032.

Horii, H. and Nemat-Nasser, S. (1983). Overall moduli of solids with microcracks: load induced anisotropy. *J. Mech. Phys. Solids* **31,** 155–171.

Kachanov, L. M. (1958). On the creep rupture time. *Izv. ANSSR, Otd. Tekhn. Nauk.* **8,** 26–31.

Kachanov, M. (1980). A continuum model medium with cracks. *J. Engng. Mech. Div. ASCE* **106,** 1039–1051.

Krajcinovic, D. (1989). Damage mechanics. *Mech. Mater.* **8,** 117–197.

Lemaitre, J. (1985). A continuous damage mechanics model for ductile fracture. *J. Engng Mater. Technol.* **107,** 83–102.

Lemaitre, J. (1987). Formulation and identification of damage kinetic constitutive equations. In *Continuum Damage Mechanics, Theory and Applications* (Edited by D. Krajcinovic and J. Lemaitre), Springer, Wien.

Lemaitre, J. and Chaboche, J. L. (1978). Aspect phenomenologique de la rupture par endommagement. *J. Mech. Appl.* **2,** 317–365.
Lemaitre, J. and Chaboche, J. L. (1985). *Mechanique des Materiaux Solides.* Dunod, Paris.
Losi, G. (1994). Numerical methods for the study of localization in elasto-plastic solids. *Mech. Res. Commun.* **3,** 94.
Mori, T. and Tanaka, K. (1973). Average stress in matrix and average elastic energy of materials with misfitting inclusions. *Acta Metal.* **21,** 571.
Murakami, S. (1988). Mechanical modelling of material damage. *J. Appl. Mech.* **55,** 280–286.
Ortiz, M. (1985). A constitutive theory for the inelastic behavior of concrete. *Mech. Mater.* **4,** 67–93.
Press, W. H., Flannery, B. P., Teukolsky, S. A. and Vetterling, W. T. (1988). *Numerical Recipes in C; the Art of Scientific Computing.* Cambridge University Press, Cambridge.
Simo, J. C. and Wu, J. W. (1987a). Strain- and stress-based continuum damage models—I. Formulation. *Int. J. Solids Structures* **23**(7), 821–840.
Simo, J. C. and Wu, J. W. (1987b). Strain- and stress-based continuum damage models—II. Computational aspects. *Int. J. Solids Structures* **23**(7), 841–869.
Smith, G. M. and Young, L. E. (1955). Ultimate theory in flexure by exponential function. *Proc. ACI* **52,** 349–359.

APPENDIX: TENSOR VALUED FUNCTIONS OF SYMMETRIC TENSORS

The problem of defining and differentiating tensor valued functions of symmetric second order tensors has been approached by Hoger (1986). Briefly, the definitions and some typical examples of this class of functions will now be recalled. Given any symmetric second order tensor **A**, one can write its decomposition in terms of the eigenvector components, namely

$$A_{ij} = \sum_{\alpha=1}^{3} \lambda_\alpha n_i^\alpha n_j^\alpha \tag{A1}$$

where the λ_αs are the three real eigenvalues and the n_i^α represents the ith component of the corresponding eigenvector. One can define any arbitrary tensor valued function of **A** as the tensor which, in a principal reference frame for **A**, is also diagonal and has as diagonal elements scalar functions of the corresponding eigenvalues of **A**. For example, the logarithm of a tensor is defined by

$$\mathrm{LOG}\,(\mathbf{A}) = \sum_{\alpha=1}^{3} \log(\lambda_\alpha) n_i^\alpha n_j^\alpha, \quad \lambda_\alpha > 0. \tag{A2}$$

Another useful tensor valued function is the step function tensor. If one considers the Heaviside step function h, defined as

$$h(x) = \begin{cases} 1, & \text{if } x \geqslant 0 \\ 0 & \text{otherwise,} \end{cases}$$

then the step function tensor is defined by

$$\mathbf{H}(\mathbf{A}) = \mathbf{H}^{+(\mathbf{A})} = \sum_{\alpha=1}^{3} h(\lambda_\alpha) n_i^\alpha n_j^\alpha. \tag{A3}$$

Using the step function tensor, one can build the projection operator $\mathbf{P}^{+(\mathbf{A})}$ based on the positive directions of the tensor **A**. This operator, which is a four-dimensional tensor, is written in component form as

$$P_{ijkl}^{+(\mathbf{A})} = H_{ik}^{+(\mathbf{A})} H_{jl}^{+(\mathbf{A})}.$$

One can show that the projection operator, when applied to the base tensor **A**, removes all of its negative eigenvalues; one can easily verify this property by writing the tensor product in the common principal reference frame of **A** and $\mathbf{P}^{+(\mathbf{A})}$. If one instead applies the operator $\mathbf{P}^{+(\mathbf{A})}$ to a tensor which does not share the same principal reference frame, then the results are somewhat different. For the sake of simplicity, consider as reference frame any principal frame of **A**; given the tensor **S** of non-vanishing components S_{ij}, the product $\mathbf{P}^{+(\mathbf{A})}:\mathbf{S}$ yields

$$\mathbf{P}^{+(\mathbf{A})}:\mathbf{S} = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{if only } \lambda_1(\mathbf{A}) \geqslant 0$$

$$\mathbf{P}^{+(\mathbf{A})}:\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{if } \lambda_1(\mathbf{A}), \lambda_2(\mathbf{A}) \geqslant 0$$

$$\mathbf{P}^{+(\mathbf{A})}:\mathbf{S} = \mathbf{S} \quad \text{if } \lambda_1(\mathbf{A}), \lambda_2(\mathbf{A}), \lambda_3(\mathbf{A}) \geqslant 0.$$

The use of the projection operator based on the critical stress difference has been previously proposed (Section 2) as a tool for implementing the yield condition in solids undergoing anisotropic damage. As defined above, this operator presents a discontinuity when the eigenvalues of the critical stress difference $\mathbf{z} = \boldsymbol{\sigma} - \mathbf{t}$ transition from negative to positive or zero, with the corresponding onset of numerical complications. In order to avoid these problems and also to have a better correspondence of the numerical model with the physical world† one can use different definitions of the projection operator based on base functions which are smoother than the Heaviside step function. Any function which goes from zero to unity in going across the origin can be used to construct such smooth operators. A good candidate is given by the operator based on the error function tensor ERFC (**A**),

$$\mathrm{ERFC}\,(\mathbf{A}) = \sum_{\alpha=1}^{3} \mathrm{erfc}\,(\lambda_\alpha, \sigma_v) n_i^\alpha n_j^\alpha \tag{A4}$$

where

$$\mathrm{erfc}\,(\lambda_\alpha, \sigma_v) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\lambda_\alpha} \mathrm{e}^{-(\xi^2/\sigma_v^2)}\,\mathrm{d}\xi. \tag{A5}$$

† As the Romans used to say "natura non facit saltus" (nature does not make jumps).